Numerical solutions for first-order ordinary differential equations

Outline

The Cauchy Problem

- First order ordinary differential equation:

$$x' = f(t, x,) \tag{1}$$

where $x' = \frac{dx}{dt}$. The general solution: $x = x(t, C),\ C \in \mathbb{R}$

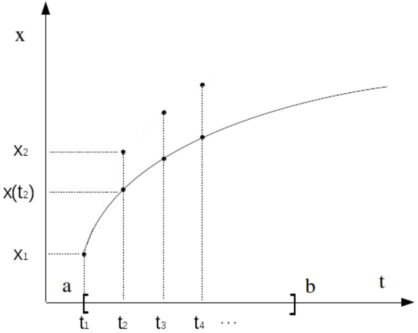
- The Cauchy problem:

$$\begin{cases} x' = f(t, x) \\ x(t_1) = x_1 \end{cases} \tag{2}$$

The particular solution: $x = x(t)$

Numerical solution for a Cauchy problem

- A **numerical (approximate) solution** of a Cauchy problem - a sequence of points following the plot of the exact (analytical) solution.



- The division: $\Delta : a = t_1 < t_2 < \cdots < t_{n+1} = b$.
  The numerical solution: the sequence of values $x_1, x_2, ..., x_{n+1}$, where $x(t_i) \simeq x_i$.
- x_1 is given by the initial condition and $x_2, ..., x_{n+1}$ may be computed by using an iterative formula of the type $x_{i+1} = F(t_i, x_i)$.

Euler's method - the formula

- $\Delta : a = t_1 < t_2 < \cdots < t_{n+1} = b$ - **equidistant** division: $t_{i+1} - t_i = h$, where $h = \frac{b-a}{n}$ is **the step of the division**.

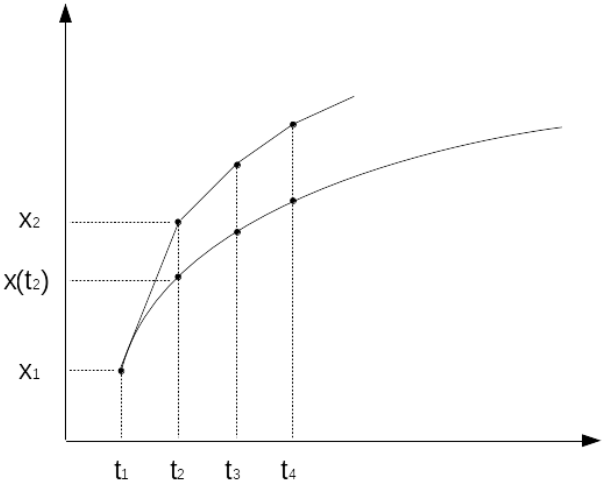
- We expand $x(t_{i+1})$ in Taylor series and keep only the first two terms of the expansion (first degree Taylor polynomial):

$$x(t_{i+1}) = x(t_i + h) \simeq x(t_i) + \frac{x'(t_i) \cdot h^1}{1!} = x(t_i) + f(t_i, x_i) \cdot h$$

- We denote by x_k the approximate value of the solution in t_k (i.e. $x_k \simeq x(t_k)$). We obtain the recurrence relation:

$$x_{i+1} = x_i + h \cdot f(t_i, x_i), \quad i = 1, 2, \cdots, n \tag{3}$$

Euler's method - geometrical interpretation



Euler's method - algorithm

Input data: Starting point (t_1, x_1) (given by the initial condition), integrating interval $[a, b]$ (where $a = t_1$), the number of subintervals n.

Output data: Approximate values $x_i,\ i = 2, 3, \cdots, n+1$ of the unknown function x in the corresponding nodes t_i of the division.

Start
$h = (b - a)/n$;
For i *from* 1 *to* n
$\quad t_{i+1} = t_i + h$
$\quad x_{i+1} = x_i + h \cdot f(t_i, x_i)$
Stop

Example - Analytical (exact) solution

Consider the Cauchy problem $\begin{cases} x' = -2 \cdot t \cdot x^2 \\ x(0) = 1 \end{cases}$. *Find the analytical solution, compute a numerical solution on the* $[0, 1]$ *interval using Euler's method and compare the two solutions.*

Analytical (exact) solution:

- We separate the variables: $\frac{dx}{dt} = -2 \cdot t \cdot x^2 \Rightarrow \frac{1}{x^2} \cdot dx = -2 \cdot t \cdot dt$ $\Rightarrow \int \frac{1}{x^2} \cdot dx = \int -2 \cdot t \cdot dt \Rightarrow -\frac{1}{x} = -t^2 - C \Rightarrow \frac{1}{x} = t^2 + C \Rightarrow x = \frac{1}{t^2+C}$
- The general solution of the differential equation is $x(t) = \frac{1}{t^2+C}$, $C \in \mathbb{R}$.
- We find C by using the initial condition: $x(0) = 1 \Rightarrow \frac{1}{0^2+C} = 1 \Rightarrow \frac{1}{C} = 1 \Rightarrow C = 1$.
- The particular solution of the Cauchy problem is $x(t) = \frac{1}{t^2+1}$.

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Numerical (approximate) solution:

- We choose the equidistant division (with the step $h = 0.1$):

$$\Delta : t_1 = 0 < t_2 = 0.1 < t_3 = 0.2 < ... < t_{11} = 1$$

  We have $f(t, x) = -2 \cdot t \cdot x^2$ and thus Euler's formula $x_{i+1} = x_i + h \cdot f(t_i, x_i)$ becomes:

$$x_{i+1} = x_i - 0.2 \cdot t_i \cdot x_i^2, \quad i = 1, 2, \cdots, 10$$

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- For $i = 2$ we compute $x_3 = x_2 - 0.2 \cdot t_2 \cdot x_2^2 = 1 - 0.2 \cdot 0.1 \cdot 1^2 = 1 - 0.02 = 0.98$.
- For $i = 3$ we compute $x_4 = x_3 - 0.2 \cdot t_3 \cdot x_3^2 = 0.98 - 0.2 \cdot 0.2 \cdot 0.98^2 = 1 - 0.02 = 0.98 - 0.04 \cdot 0.9604 = 0.98 - 0.038416 = 0.9416$ and the computations may continue in the same manner ($i = 4, 5, \cdots, 10$).

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Example - Comparison between the analytical solution and the numerical one

- The difference (in absolute value) between the exact value $x(t_i)$ and the corresponding approximation x_i is in fact the error associated to the approximation: $\varepsilon_i = |x(t_i) - x_i|$.
- Since the exact solution is $x(t) = \frac{1}{t^2+1}$, the exact values $x(t_i)$ corresponding to the previously computed approximations x_i are: $x(t_1) = \frac{1}{t_1^2+1} = \frac{1}{0^2+1} = 1$, $x(t_2) = \frac{1}{t_2^2+1} = \frac{1}{0.1^2+1} = \frac{1}{1.01} \simeq 0.99$, $x(t_3) = \frac{1}{t_3^2+1} = \frac{1}{0.2^2+1} = \frac{1}{1.04} \simeq 0.961$, $x(t_4) = \frac{1}{t_4^2+1} = \frac{1}{0.3^2+1} = \frac{1}{1.09} \simeq 0.917$.
- The results corresponding to the first three steps in Euler's method are:

t_i	Exact sol. $x(t_i)$	Numerical sol. x_i	Error $\varepsilon_i = \lvert x(t_i) - x_i \rvert$
$t_1 = 0$	$x(t_1) = 1$	$x_1 = 1$	$\varepsilon_1 = \lvert 1 - 1 \rvert = 0$
$t_2 = 0.1$	$x(t_2) \simeq 0.990...$	$x_2 = 1$	$\varepsilon_2 \simeq \lvert 0.99 - 1 \rvert \simeq 0.01$
$t_3 = 0.2$	$x(t_3) \simeq 0.961...$	$x_3 = 0.98$	$\varepsilon_3 \simeq \lvert 0.961 - 0.98 \rvert \simeq 0.02$
$t_4 = 0.3$	$x(t_4) \simeq 0.917...$	$x_4 \simeq 0.941...$	$\varepsilon_4 \simeq \lvert 0.917 - 0.941 \rvert \simeq 0.03$

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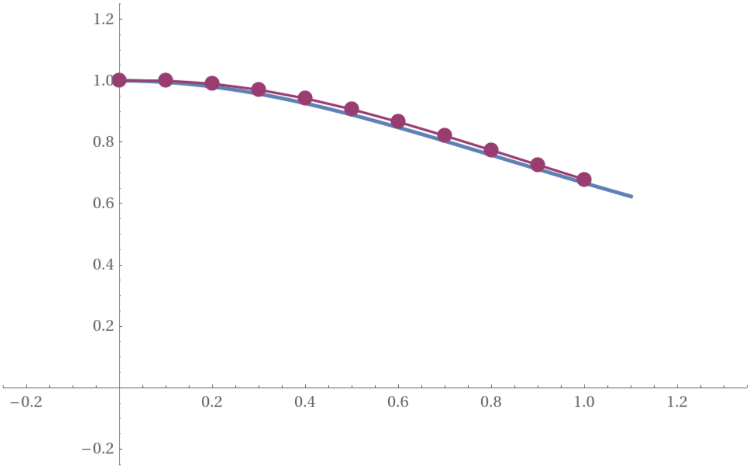

Remarks:

- Due to the nature of any method of Euler's type, which involves a truncation of the Taylor series expansion, the value of the approximation x_i presents an inherent error. In the case of Euler's method the errors are rather large and, even though they can be reduced by choosing a smaller step size, Euler's method is not considered as a practical method.
- Euler's method is a *single-step* method, meaning that x_{i+1} is computed as a function of x_i only, in contrast to *multistep* methods which compute x_{i+1} as a function of not only x_i but also x_{i-1}, x_{i-2} ..., thus obtaining a more precise approximation.

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Remarks:

- Another way to obtain a better approximation is to take into account more terms in the Taylor series expansion of $x(t_{i+1}) = x(t_i + h)$. Thus, keeping the first three terms we obtain (a second degree Taylor polynomial):

$$x(t_i + h) \simeq x(t_i) + \frac{x'(t_i)}{1!} \cdot h + \frac{x''(t_i)}{2!} \cdot h^2$$

  Here $x''(t_i) = \frac{d}{dt}(x'(t_i)) = \frac{d}{dt}(f(t_i, x(t_i))) = \frac{\partial f}{\partial t}(t_i, x_i) + \frac{\partial f}{\partial x}(t_i, x_i) \cdot \frac{dx}{dt} =$
  $= \frac{\partial f}{\partial t}(t_i, x_i) + \frac{\partial f}{\partial x}(t_i, x_i) \cdot f(t_i, x_i)$.

  It follows that the Taylor series expansion of $x(t_i + h)$ is:

$$x(t_i + h) \simeq x_i + f(t_i, x_i) \cdot h + \frac{1}{2}\left(\frac{\partial f}{\partial t}(t_i, x_i) + \frac{\partial f}{\partial x}(t_i, x_i) \cdot f(t_i, x_i)\right) h^2. \tag{4}$$

Runge-Kutta-type methods - introduction

Runge-Kutta methods of order r are a family of numerical methods based on the folowing formula:

$$x_{i+1} = x_i + \sum_{j=1}^{r} c_j \cdot k_j \tag{5}$$

where:

$$k_1 = h \cdot f(t_i, x_i), \quad k_j = h \cdot f(t_i + \alpha_j \cdot h, \; x_i + \sum_{s=1}^{j-1} \beta_{js} \cdot k_s), \; j = 2, ..., r$$

The constants c_j, α_j and β_{js} are determined by imposing the following condition:

• **The coefficients of the powers of h from the Taylor series expansion of x_{i+1} given by (5) must coincide with the corresponding coefficients from the Taylor series expansion (4) of $x(t_i + h)$.**

The methods defined by the formulas (5) are called **Runge-Kutta methods of order** r.

Runge-Kutta-type methods - The case r=1

For $r = 1$ the formula (5) becomes:

$$x_{i+1} = x_i + c_1 \cdot k_1, \quad k_1 = h \cdot f(t_i, x_i) \quad \Rightarrow \quad x_{i+1} = x_i + c_1 \cdot h \cdot f(t_i, x_i) \qquad (*)$$

Basically in this case there is no need for a Taylor series expansion of x_{i+1} (since it already appears as a first degree polynomial in h).

On the other hand the corresponding Taylor series expansion of $x(t_i + h)$ is (the first degree Taylor polynomial):

$$x(t_i + h) \simeq x(t_i) + h \cdot x'(t_i) = x_i + h \cdot f(t_i, x_i) \qquad (**)$$

Comparing the above expressions of x_{i+1} $(*)$ and $x(t_i + h)$ $(**)$ we remark that in both expressions the coefficient of h^0 (the "free term" with respect to h) is x_i. The coefficient of h^1 in $(*)$ is $c_1 \cdot f(t_i, x_i)$, while in $(**)$ it is $f(t_i, x_i)$. Since the two coefficients must be equal, it follows that $c_1 = 1$ and in fact **the first order Runge-Kutta method** coincides with Euler's method: $x(t_i + h) = x_i + h \cdot f(t_i, x_i)$.

Runge-Kutta-type methods - The case r=2

For $r = 2$ the formula (5) becomes:

$$x_{i+1} = x_i + c_1 \cdot k_1 + c_2 \cdot k_2, \quad k_1 = h \cdot f(t_i, x_i), \; k_2 = h \cdot f(t_i + \alpha_2 \cdot h, x_i + \beta_{21} \cdot k_1).$$

Replacing k_1 and k_2 we obtain:

$$x_{i+1} = x_i + c_1 \cdot h \cdot f(t_i, x_i) + c_2 \cdot h \cdot f(t_i + \alpha_2 \cdot h, x_i + \beta_{21} \cdot h \cdot f(t_i, x_i)). \tag{6}$$

We recall that a function of the type $f(a + h_1, b + h_2)$ may be expanded in Taylor series:

$$f(a + h_1, b + h_2) \simeq f(a, b) + h_1 \cdot \frac{\partial f}{\partial t}(a, b) + h_2 \cdot \frac{\partial f}{\partial x}(a, b)$$

For $a = t_i$, $b = x_i$, $h_1 = \alpha_2 \cdot h$ and $h_2 = \beta_{21} \cdot h \cdot f(t_i, x_i)$ we obtain the expansion:

$$f(t_i + \alpha_2 \cdot h, x_i + \beta_{js} \cdot h \cdot f(t_i, x_i)) \simeq f(t_i, x_i) + \alpha_2 \cdot h \cdot \frac{\partial f}{\partial t}(t_i, x_i) + \beta_{21} \cdot h \cdot f(t_i, x_i) \cdot \frac{\partial f}{\partial x}(t_i, x_i).$$

Replacing this expansion in (6) we obtain:

$$x_{i+1} \simeq x_i + c_1 \cdot h \cdot f(t_i, x_i) + c_2 \cdot h \cdot \left(f(t_i, x_i) + \alpha_2 \cdot h \cdot \frac{\partial f}{\partial t}(t_i, x_i) + \beta_{21} \cdot h \cdot f(t_i, x_i) \cdot \frac{\partial f}{\partial x}(t_i, x_i) \right)$$

It follows that the Taylor series expansion of x_{i+1} corresponding to the (5) formula is:

$$x_{i+1} \simeq x_i + (c_1 + c_2) \cdot f(t_i, x_i) \cdot h + \left(c_2 \cdot \alpha_2 \cdot \frac{\partial f}{\partial t}(t_i, x_i) + c_2 \cdot \beta_{21} \cdot f(t_i, x_i) \cdot \frac{\partial f}{\partial x}(t_i, x_i) \right) \cdot h^2 \tag{7}$$

Runge-Kutta-type methods - The case r=2 (continued)

Thus in order to find the constants c_1, c_2, α_2 and β_{21} we will compare the coefficients of the powers of h from the Taylor series expansion of x_{i+1} (7) corresponding to the Runge-Kutta formula:

$$x_{i+1} \simeq x_i + (c_1 + c_2) \cdot f(t_i, x_i) \cdot h + \left(c_2 \cdot \alpha_2 \cdot \frac{\partial f}{\partial t}(t_i, x_i) + c_2 \cdot \beta_{21} \cdot f(t_i, x_i) \cdot \frac{\partial f}{\partial x}(t_i, x_i) \right) \cdot h^2$$

and the coefficients of the powers of h from the Taylor series expansion of $x(t_i + h)$ (4):

$$x(t_i + h) \simeq x_i + f(t_i, x_i) \cdot h + \frac{1}{2} \left(\frac{\partial f}{\partial t}(t_i, x_i) + \frac{\partial f}{\partial x}(t_i, x_i) \cdot f(t_i, x_i) \right) h^2.$$

- The coefficients of h^0 are the same in both expansions, namely x_i.
- The coefficient of h^1 în (7) is $(c_1 + c_2) \cdot f(t_i, x_i)$ while the same coefficient in (4) is $f(t_i, x_i)$, which means that $c_1 + c_2 = 1$.
- The coefficient of h^2 in (7) contains $c_2 \cdot \alpha_2$ as the coefficient of $\frac{\partial f}{\partial t}(t_i, x_i)$ and $c_2 \cdot \beta_{21}$ as the coefficient of $f(t_i, x_i) \cdot \frac{\partial f}{\partial x}(t_i, x_i)$. In (4) the corresponding coefficients are both equal to $\frac{1}{2}$, which means that $c_2 \cdot \alpha_2 = \frac{1}{2}$ and $c_2 \cdot \beta_{21} = \frac{1}{2}$.

Runge-Kutta-type methods - The case r=2 (continued)

Hence we obtain the system of equations:

$$\begin{cases} c_1 + c_2 = 1 \\ c_2 \cdot \alpha_2 = \frac{1}{2} \\ c_2 \cdot \beta_{21} = \frac{1}{2} \end{cases}$$

Obviously this system of 3 equations in 4 unknowns does not have a unique solution, which means that there is no unique Runge-Kutta method of second order, and this is true for any Runge-Kutta method of order $r,\ r > 1$.

One of the well-known variants of the second order Runge-Kutta method corresponds to the values $c_1 = c_2 = \frac{1}{2}$, $\alpha_2 = \beta_{21} = 1$:

$$x_{i+1} = x_i + \frac{h}{2}\left[f(t_i, x_i) + f(t_i + h, x_i + h \cdot f(t_i, x_i))\right]. \tag{8}$$

Runge-Kutta-type methods - The case r=4

One of the most widely used methods to find numerical solutions of ordinary differential equations is the fourth order Runge-Kutta method. The computation of the coefficients (too long to be included here) is performed in the same way as for the previous cases (Exercise!). A variant of the fourth order Runge-Kutta method is:

$$x_{i+1} = x_i + \frac{k_1 + 2\cdot k_2 + 2\cdot k_3 + k_4}{6}, \quad i = 1, 2, \cdots, n \tag{9}$$

unde

$$k_1 = h\cdot f(t_i, x_i),\ k_2 = h\cdot f(t_i+\frac{h}{2},\ x_i+\frac{k_1}{2}),\ k_3 = h\cdot f(t_i+\frac{h}{2},\ x_i+\frac{k_2}{2}),\ k_4 = h\cdot f(t_i+h,\ x_i+k_3).$$

Algorithm for the fourth order Runge-Kutta method:

Input data: The starting point (t_1, x_1) (given by the initial condition), the interval of integration $[a, b]$ (where $a = t_1$), the number of subdivisions n .

Output data: The approximate values $x_i,\ i = 2, 3, \cdots, n+1$ of the unknown function x in the corresponding nodes t_i of the division.

Start
$h = (b - a)/n;$
Pentru i de la 1 *la n*
$$t_{i+1} = t_i + h$$
$$k_1 = h \cdot f(t_i, x_i)$$
$$k_2 = h \cdot f(t_i + \frac{h}{2}$$
$$k_3 = h \cdot f(t_i + \frac{h}{2},\ x_i + \frac{k_2}{2})$$
$$k_4 = h \cdot f(t_i + h,\ x_i + k_3)$$
$$x_{i+1} = x_i + \frac{k_1 + 2 \cdot k_2 + 2 \cdot k_3 + k_4}{6}$$
Stop

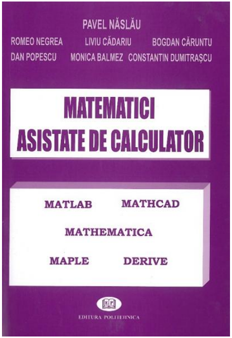
Exercise

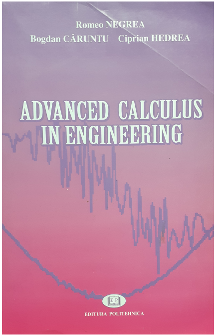
We consider the Cauchy problem $\begin{cases} y' = 2 \cdot x \cdot y \\ y(0) = -1 \end{cases}$. Find the analytical (exact) solution of the problem, compute a numerical solution using Euler's method (the first three steps only) and compare the two solutions.

References

Matematici asistate de calculator. Matlab, Mathcad, Mathematica, Maple, Derive *Pavel Naslau, Romeo Negrea, Liviu Cadariu, Bogdan Caruntu, Dan Popescu, Monica Balmez, Constantin Dumitrascu*, Editura Politehnica, Timisoara, 2007.

Advanced Calculus in Engineering, *Romeo Negrea, Bogdan Caruntu, Ciprian Hedrea*, Editura Politehnica, Timisoara, 2009.





Class evaluation form - CV

Week 7

Edit

Lecture 6 204.6KB PDF document

Edit

Wednesday 06.04.2022, 15.00, Laboratory hall (Department of Mathematics, Rectorate building)

Class evaluation form

Edit

With the aim of improving the course, please take a few moments to fill this (short and anonymous) survey.

Thank you for your attention!